Local Energy Estimates for Wave Equations with Degenerate Trapping

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Abstract

Local energy estimates are a robust measure for quantifying the energy decay for wave equations, and they have become a fundamental tool in the study of wave equations on asymptotically flat spacetimes. We seek to obtain a localized energy estimate for wave equations in the presence of degenerate trapping, by studying a warped product manifold whose generating function has an inflection point, which corresponds to trapped rays. Based on the preceding work of Christianson and Metcalfe in ”Sharp local smoothing for warped product manifolds with smooth inflection transmission”, which is an analog of local energy estimates for wave equations, it is expected that the result of this study will provide a second explicit example of a case where local energy estimates hold but with a sharp algebraic loss. In this paper, we first establish an exterior estimate that provides the local energy estimate away from the trapped sets produced by the geometry. Then we use a refinement of the exterior estimate to establish a low frequency estimate and a high frequency estimate with respect to their different frequency regimes. The project has the potential of expanding the frontier of our knowledge in local energy estimates for wave equations, and we hope that the specific case of local energy estimates with degenerate trapping that we examine will provide a roadmap for extending the local energy estimates techniques to more general geometries.

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Contents

Introduction

A robust measure of decay and dispersion for the wave equation is provided by the localized energy estimates, and they have become a fundamental tool in the study of wave behaviors on asymptotically flat spacetime. Conceptually, the local energy estimates can be proved by commuting the wave operator with a carefully chosen multiplier so that upon integration by parts all the resulting terms are positive. This method is first developed by Cathleen Morawetz in the context of the Klein-Gordon equation in [1] and it was generalized by several scholars.(See [2],[3]). We will present both in details in Chapter 1 and 2, respectively. The local energy estimates have also been used to prove other types of estimates such as Strichartz estimates [4],[5] and pointwise decay estimates [6], [7].

On Minkowski space, it has been proved that solutions to wave equations enjoy a conserved energy, meaning that the energy is the same for all times (proof provided in Chapter 1). Therefore, the energy does not decay. However, if we choose to consider a compact set, then we expect that energy carried by the waves decays within the region as the waves will disperse and eventually leave the region.

In general relativity, which postulates that the universe is a curved spacetime and the curvature of spacetime is directly related to gravity, understanding wave equations leads to critical insights about waves near black holes and other physical objects that cause the curvature of the space to be significant. Trapping is said to occur when a background geometry (e.g. black hole backgrounds) allows a null geodesic, and hence energy from a disturbance, to remain within a bounded region for all time. This occurs, e.g., on the photon sphere in the Schwarzschild spacetime from general relativity. Local energy estimates are sensitive to such geodesic trappings which are known to be an obstruction in quantifying local energy decay. Their presences necessitate a loss in the estimates when compared to the estimates available in Minkowski space. See [8], [9].

Many works have shown that local energy decay can be recovered with a minimal loss when the trapped geodesics are sufficiently unstable, see e.g., [4], [10], [11], while stable trapping features elimination of most energy decay [12]. Examples of in-between scenarios remained unknown until the first explicit example is proved in [13] based on the proceeding work in [14]. In the study, the authors derived an estimate of energy decay that has an sharp algebraic loss. This is accomplished by studying a warped product manifold, which is a higher dimensional analog of a surface of revolution, whose generating function has a degenerate (i.e. flattened out) minimum.

Similar estimates also exist for the Schrödinger equation. In [14], the authors studied the Schrödinger equation with a family of surfaces of revolution, each with a single periodic geodesic which is degenerately unstable and proved the first example of local smoothing (which is an analogy of local energy for the wave equation) with a sharp algebraic loss. In [15], the authors considered a family of rotationally symmetric, asymptotically Euclidean manifolds with two trapped sets, one of which is unstable and one of which is semistable created by a inflection point in the generating function, and showed a local smoothing estimate with a loss that depends on the degeneracy.

In this paper, we study the wave equation on the same geometric background as in [15]. We will utilize techniques developed in [2], [13]. We will first obtain a local energy estimate exterior to a ball with sufficiently large radius. Then we establish an estimate for a low frequency regime and an estimate for a high frequency regime respectively. We call them "high frequency estimate" and "low frequency" estimate" for the reason that if we restrict to sufficiently high time-frequency and sufficiently low time-frequency, we will be able to bootstrap the error terms.

Chapter 1

Conservation of Energy and Morawetz Estimates

We study the homogeneous wave equation $\Box u(t,x) = 0$. Here the wave operator is given by the d'Alembertian. The flat Minkowski metric in (1+3) dimensions is given by $m = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$. In particular, on $\mathbb{R}_+ \times \mathbb{R}^3$ with the flat Minkowski metric, the wave operator is given by

$$\Box u = \partial_t^2 u - \Delta u, \quad \Delta u = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} u.$$

The energy of a solution to the wave equation is defined as

$$E[u](t) = \frac{1}{2} \int (\partial_t u)^2 + |\nabla u|^2 \; dx. \tag{1.1}$$

Here we shall use the L^2 norm, which is defined as

$$||f(x)||_{L^2} = \Big(\int |f(x)|^2 \; dx \Big)^{1/2} \tag{1.2}$$

and $u'(t,x)$ is the spacetime gradient of $u(t,x)$, which is given by

$$u'(t,x) = (\partial_t u, \nabla u). \tag{1.3}$$

On Minkowski space, solutions to wave equation enjoy a conserved energy, meaning that the energy is the same for all times. I.e., for all t, we have

$$E[u](t) = E[u](0).$$

We shall obtain:

Theorem 1.0.1. *Suppose* $u(t,x) \in C^2$ *solves the homogeneous wave equation* $\Box u = 0$ *on* $(0,T) \times \mathbb{R}^3$, *and* u *vanishes for sufficiently large* $|x|$. *Then for any* $t \in (0,T)$,

$$\left\|u'(t,\cdot)\right\|_{L^2}^2 = \left\|u'(0,\cdot)\right\|_{L^2}^2 .$$

The conservation of energy of the wave equation is a key fact that we will use in obtaining local energy estimates later in this chapter.

Proof. The first part of the proof involves a special case of integration by parts. The finite speed of propogation of wave equation implies that u vanishes for sufficiently large $|x|$ for any fixed t if the initial data are compactly supported, so when we move (one order of) spatial derivative onto the other derivative, the evalutation at the end points for the antiderivative is essentially 0. Explicitly, this looks like

$$\int u \; dv = -\int v \; du.$$

Also note that ∂_t and ∂_x simply commute since they are independent. In addition, we also make use of the reversed chain rule in this proof and throughout this paper, which is

$$u \; du = \frac{1}{2} d(u^2).$$

We multiply $\Box u$ by $\partial_t u$ and integrate by parts. By taking advantage of the fact that $\Box u = 0$ and using the reversed chain rule, we have

$$\begin{aligned}
0 =& \int_0^T \int_{\mathbb{R}^3} \Box u \partial_t u \; dxdt \\
=& \int_0^T \int_{\mathbb{R}^3} (\partial_t^2 u - \partial_{x_1}^2 u - \partial_{x_2}^2 u - \partial_{x_3}^2 u)\partial_t u \; dxdt \\
=& \int_0^T \int_{\mathbb{R}^3} \partial_t^2 u \partial_t u + \partial_{x_1} u \partial_t \partial_{x_1} u + \partial_{x_2} u \partial_t \partial_{x_2} u + \partial_{x_3} u \partial_t \partial_{x_3} u \; dxdt \\
=& \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \partial_t (\partial_t u)^2 + \partial_t (\partial_{x_1} u)^2 + \partial_t (\partial_{x_2} u)^2 + \partial_t (\partial_{x_3} u)^2 \; dxdt \\
=& \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \partial_t \Big((\partial_t u)^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2 + (\partial_{x_3} u)^2 \Big) \; dxdt.
\end{aligned}$$

Applying Fundamental Theorem of Calculus, this becomes

$$\begin{aligned}
0 =& \frac{1}{2} \int_{\mathbb{R}^3} \left((\partial_t u)^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2 + (\partial_{x_3} u)^2 \right) \; dx \Big|_0^T \\
=& \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t u)^2 + |\nabla u|^2 \; dx \Big|_0^T .
\end{aligned}$$

Using the L^2 norm defined in (1.2), we have

$$\left\|u'(t,\cdot)\right\|_{L^2}^2 - \left\|u'(0,\cdot)\right\|_{L^2}^2 = 0 \tag{1.4}$$

as desired. $\square$

Now that we have conservation of energy, we cannot obtain meaningful estimates that measure energy decay by simply integrating over time. The idea here is that we choose to consider a compact set, then we expect the energy within the set to decay over time, and thus we could do integration within that region over time.

We first introduce the notations used in this section. Let $r = |x|$, $x \in \mathbb{R}^3$, and $\partial_r u = \frac{x}{r} \cdot \nabla u$. Here we decompose the gradient into angular and radial derivatives.

$$\nabla u = \frac{x}{r}\partial_r u + \not\nabla u. \tag{1.5}$$

Here $\not\nabla$ denotes the angular derivative.

Lemma 1.0.2. *The resulting two parts from the decomposition are orthogonal to each other. I.e.*

$$\frac{x}{r}\partial_r u \cdot \not\nabla u = 0.$$

Proof. Using the decomposition, it directly follows that

$$\begin{aligned}
\frac{x}{r}\partial_r u \cdot \not\nabla u =& \frac{x}{r}\partial_r u \cdot (\nabla u - \frac{x}{r}\partial_r u) \\
=& \frac{x}{r}\partial_r u \cdot \nabla u - \frac{x}{r}\partial_r u \cdot \frac{x}{r}\partial_r u \\
=& (\partial_r u)^2 - (\partial_r u)^2 \\
=& 0.
\end{aligned}$$

□

Lemma 1.0.3. $|\nabla u|^2 = (\partial_r u)^2 + |\not\nabla u|^2$.

Proof. This follows directly from the decomposition. Notice that when we square the right hand side, the term from cross multiplication is 0 by Lemma 1.0.2. □

Lemma 1.0.4. $\nabla \partial_r u = \partial_r \nabla u + \frac{1}{r}\not\nabla u$.

Proof. We prove the statement component-wise. Expanding the right hand side, we have

$$\begin{aligned}
\nabla_j \partial_r u =& \sum_{i=1}^{3} \nabla_j \Big(\frac{x_i}{r} \partial_i u \Big) \\
=& \sum_{i=1}^{3} \Big(\nabla_j (\frac{x_i}{r}) \partial_i u + \frac{x_i}{r} \partial_i \nabla_j u \Big) \\
=& \sum_{i=1}^{3} \Big(\nabla_j (\frac{x_i}{r}) \partial_i u \Big) + \partial_r \nabla_j u \\
=& \sum_{i=1}^{3} \Big((\frac{\delta_{ij}}{r} - \frac{x_i}{r^2} \frac{x_j}{r}) \partial_i u \Big) + \partial_r \nabla_j u \\
=& \frac{1}{r} \Big(\partial_j u - \frac{x_j}{r} \partial_r u \Big) + \partial_r \nabla_j u \\
=& \frac{1}{r} \not\nabla_j u + \partial_r \nabla_j u
\end{aligned}$$

where the last step follows directly from the decomposition in (1.5). □

Now we are ready to state and prove the Morawetz estimate, which is an important estimate for solutions of wave equations on Minkowski space (first explored in [1]).

Theorem 1.0.5. *Suppose that $\Box u = 0$ on $\mathbb{R}_+ \times \mathbb{R}^3$, and that for every t, $u(t,x)$ vanishes for sufficiently large $|x|$. Then, for any $T > 0$,*

$$\int_0^T \int_{\mathbb{R}^3} \frac{|\not\nabla u|^2}{|x|} \, dxdt \lesssim E[u](0), \tag{1.6}$$

where the implicit constant is independent of T.

Proof. Since $\Box u = 0$, consider

$$0 = \int_0^T \int_{\mathbb{R}^3} \Box u \partial_r u \, dxdt.$$

Expanding the right hand side and integrating by parts, we have

$$0 = \int_{\mathbb{R}^3} \partial_t u \partial_r u \, dx \Big|_0^T - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \partial_r (\partial_t u)^2 \, dxdt - \int_0^T \int_{\mathbb{R}^3} \Delta u \partial_r u \, dxdt. \tag{1.7}$$

Direct calculation gives

$$\begin{aligned}
0 = \int_{\mathbb{R}^3} \partial_t u \partial_r u \, dx \Big|_0^T - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \partial_r \Big((\partial_r u)^2 - |\nabla u|^2 \Big) \, dxdt - & \int_0^T \int_{\mathbb{R}^3} \nabla \cdot (\nabla u \partial_r u) \, dxdt \\
+ & \int_0^T \int_{\mathbb{R}^3} \frac{1}{r} |\not\nabla u|^2 \, dxdt
\end{aligned} \tag{1.8}$$

Since u vanishes at infinity, by the Divergence Theorem, the third term is 0. Focusing on the second term, and recalling that $\partial_r u = \frac{x}{r} \cdot \nabla u$, direct calculation shows

$$\frac{1}{2} \nabla \cdot \Big[\frac{x}{r} \Big((\partial_r u)^2 - |\nabla u|^2 \Big) \Big] = \frac{1}{2} \Big(\nabla \cdot \frac{x}{r} \Big) \Big((\partial_r u)^2 - |\nabla u|^2 \Big) + \frac{1}{2} \partial_r \Big((\partial_t u)^2 - |\nabla u|^2 \Big) \tag{1.9}$$

Using what we have from (1.9) and collecting like terms, (1.8) becomes

$$0 = \int_{\mathbb{R}^3} \partial_t u \partial_r u \ dx\Big|_0^T + \int_0^T \int_{\mathbb{R}^3} \frac{1}{r}\Big((\partial_t u)^2 - |\nabla u|^2\Big) \ dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{1}{r}|\not\nabla u|^2 \ dxdt. \tag{1.10}$$

We want to cancel out the second term in (1.10), and this can be achieved by modifying the multiplier. We now multiply $\Box u$ by $\frac{u}{r}$ and integrate by parts.

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^3} \Box u \frac{u}{r} \ dxdt \\ &= \int_{\mathbb{R}^3} \frac{u}{r}\partial_t u \ dx\Big|_0^T - \int_0^T \int_{\mathbb{R}^3} \frac{1}{r}(\partial_t u)^2 \ dxdt - \int_0^T \int_{\mathbb{R}^3} \frac{u}{r^2}\partial_r u \ dxdt \\ &\quad + \int_0^T \int_{\mathbb{R}^3} \frac{1}{r}(\partial_r u)^2 \ dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{1}{r}|\not\nabla u|^2 \ dxdt. \end{aligned} \tag{1.11}$$

Following similar procedures, we have

$$0 = \int_{\mathbb{R}^3} \frac{u}{r}\partial_t u \ dx\Big|_0^T - \int_0^T \int_{\mathbb{R}^3} \frac{1}{r}\Big((\partial_t u)^2 - |\nabla u|^2\Big) \ dxdt - \int_0^T \int_{\mathbb{R}^3} \frac{u}{r^2}\partial_r u \ dxdt. \tag{1.12}$$

Adding the results from the two multipliers together, we obtain

$$0 = \int_{\mathbb{R}^3} \partial_t u \partial_r u \ dx\Big|_0^T + \int_{\mathbb{R}^3} \frac{u}{r}\partial_t u \ dx\Big|_0^T + \int_0^T \int_{\mathbb{R}^3} \frac{1}{r}|\not\nabla u|^2 \ dxdt - \int_0^T \int_{\mathbb{R}^3} \frac{u}{r^2}\partial_r u \ dxdt. \tag{1.13}$$

Focusing on the last term in (1.13), we first switch to polar co ordinates and use the reversed chain rule. Then we apply the Fundamental Theorem of Calculus. Since u vanishes at infinity, it follows that

$$-\int_0^T \int_{\mathbb{R}^3} \frac{u}{r^2}\partial_r u \ dxdt = -\frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \partial_r(u^2) \ drd\theta dt = \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} u^2(t,0) \ dt \tag{1.14}$$

Now we are left to show that

$$\int_{\mathbb{R}^3} \partial_t u \partial_r u \ dx\Big|_0^T + \int_{\mathbb{R}^3} \frac{u}{r}\partial_t u \ dx\Big|_0^T \lesssim E[u](0). \tag{1.15}$$

Applying the Cauchy-Schwarz inequality for the first term in (1.15), we see that

$$\Big|\int_{\mathbb{R}^3} \partial_t u \partial_r u \ dx\Big| \leq \|\partial_t u\|_{L^2}\|\partial_r u\|_{L^2} \leq \|\partial_t u\|_{L^2}\|\nabla u\|_{L^2} \leq \big\|u'(t,\cdot)\big\|^2_{L^2}. \tag{1.16}$$

The second inequality follows from the fact that

$$\Big|\frac{x}{r}\cdot\nabla u\Big| \leq \Big|\frac{x}{r}\Big||\nabla u| \lesssim |\nabla u|.$$

Due to the fact that the energy is conserved, the quantity on the left hand side of (1.16) is bounded by $E[u](0)$.

We apply the Hardy inequality to bound the second term in (1.15) by $E[u](0)$. To achieve this, we first switch to spherical coordinates where $dx = r^2 dr d(\theta,\phi)$, and then multiply by $\partial_r r$ to move the derivative onto u using integration by parts. Specifically, the steps are

$$\begin{aligned}
\left\|\frac{u}{r}\right\|_{L^2}^2 &= \int_{\mathbb{R}^3} \frac{1}{r^2} u^2 \; dx \\
&= \int \int u^2 \; drd(\theta, \phi) \\
&= \int \int u^2 \partial_r r \; drd(\theta, \phi) \\
&= -\int \int \partial_r(u^2) r \; drd(\theta, \phi) \\
&= -\int \int 2u\partial_r ur \; drd(\theta, \phi) \\
&= -\int 2u\partial_r u \cdot \frac{1}{r} \; dx \\
&\lesssim \left\|\frac{u}{r}\right\|_{L^2} \|\partial_r u\|_{L^2} .
\end{aligned}$$

Thus, we can establish

$$\left\|\frac{u}{r}\right\|_{L^2} \leq \|\partial_r u\|_{L^2} \lesssim \|\nabla u\|_{L^2} . \tag{1.17}$$

Using each of the analysis, we now have everything bounded uniformly by $E[u](0)$, as desired.

$\square$

Chapter 2

Generalized Local Energy Estimates

In this chapter, we want to establish a generalized local energy estimate. (See [16],[3].) Like what was shown in the previous chapter, here we use the multiplier method, which involves multiplying $\Box u$ by a cleverly chosen multiplier, taking advantage of the fact that $\Box u = 0$, and integrating by parts. Later in this paper, this method will be modified and applied to a more general geometry with trapped null geodesics. We continue the notations used in the previous chapter.

Theorem 2.0.1. *If $\Box u = 0$, and for every t, u vanishes for large $|x|$, then*

$$\sup_R \Big\{ R^{-1/2} \|u'\|_{L^2_t L^2_x([0,T]\times|x|\leq R)} + R^{-3/2} ||u||_{L^2_t L^2_x([0,T]\times|x|\leq R)} \Big\} \lesssim ||u'(0,\cdot)||_{L^2}.$$

Remark 1. *This is a generalization of the Morawetz estimate as instead of just having the $\not\nabla u$ term, it also measures the decay of $\partial_t u$ and ∇u.*

Proof. First, we multiply by $f(r)\partial_r u$. Here f is a function of r, and the necessary properties of f will be shown later in the proof.

Integrating by parts, we have

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^3} \Box u f(r)\partial_r u \; dxdt \\
&= \int_{\mathbb{R}^3} \partial_t u f(r)\partial_r u \; dx\Big|_0^T - \int_0^T \int_{\mathbb{R}^3} \partial_t u \partial_t \big(f(r)\partial_r u\big) \; dxdt + \int_0^T \int_{\mathbb{R}^3} \partial_{x_1} u \partial_{x_1}\big(f(r)\partial_r u\big) \; dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \partial_{x_2} u \partial_{x_2}\big(f(r)\partial_r u\big) \; dxdt + \int_0^T \int_{\mathbb{R}^3} \partial_{x_3} u \partial_{x_3}\big(f(r)\partial_r u\big) \; dxdt \\
&= \int_{\mathbb{R}^3} \partial_t u f(r)\partial_r u \; dx\Big|_0^T - \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} f(r)\partial_r(\partial_t u)^2 \; dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \partial_{x_1} u\Big(f'(r)\cdot\frac{x_1}{r}\partial_r u + f(r)\partial_{x_1}\partial_r u\Big) \; dxdt + \int_0^T \int_{\mathbb{R}^3} \partial_{x_2} u\Big(f'(r)\cdot\frac{x_2}{r}\partial_r u + f(r)\partial_{x_2}\partial_r u\Big) \; dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \partial_{x_3} u\Big(f'(r)\cdot\frac{x_3}{r}\partial_r u + f(r)\partial_{x_3}\partial_r u\Big) \; dxdt.
\end{aligned}
\tag{2.1}
$$

Direct calculation using Lemma 1.0.2, 1.0.3, 1.0.4 in Chapter 1 to commute the derivatives gives

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^3} \partial_t u f(r)\partial_r u \; dx\Big|_0^T - \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} f(r)\partial_r(\partial_t u)^2 \; dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \partial_{x_1} u \partial_{x_1}\Big(f'(r)\cdot\frac{x_1}{r}\partial_r u + f(r)\big(\partial_r \nabla_1 u + \frac{1}{r}\not\nabla_1 u\big)\Big) \; dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \partial_{x_2} u \partial_{x_2}\Big(f'(r)\cdot\frac{x_2}{r}\partial_r u + f(r)\big(\partial_r \nabla_2 u + \frac{1}{r}\not\nabla_2 u\big)\Big) \; dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \partial_{x_3} u \partial_{x_3}\Big(f'(r)\cdot\frac{x_3}{r}\partial_r u + f(r)\big(\partial_r \nabla_3 u + \frac{1}{r}\not\nabla_3 u\big)\Big) \; dxdt \\
&= \int_{\mathbb{R}^3} \partial_t u f(r)\partial_r u \; dx\Big|_0^T - \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} f(r)\partial_r(\partial_t u)^2 \; dxdt + \int_0^T \int_{\mathbb{R}^3} f'(r)\partial_r u \nabla u\cdot\frac{x}{r} \; dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} f(r)\partial_r|\nabla u|^2 + f(r)\frac{1}{r}|\not\nabla u|^2 \; dxdt \\
&= \int \partial_t u(f(r)\partial_r u) \; dx\Big|_0^T + \int_0^T \int_{\mathbb{R}^3} f'(r)(\partial_r u)^2 \; dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r}|\not\nabla u|^2 \; dxdt \\
&\quad - \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} f(r)\partial_r\big((\partial_t u)^2 - |\nabla u|^2\big) \; dxdt.
\end{aligned}
\tag{2.2}
$$

Recalling that $\partial_r u = \frac{x}{r}\cdot\nabla u$, we can integrate the last term in (2.2) by parts further.

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} f(r)\partial_r\Big((\partial_t u)^2 - |\nabla u|^2\Big) \; dxdt \\
&= \int_0^T \int_{\mathbb{R}^3} f(r)\frac{x}{r}\cdot\nabla\Big((\partial_t u)^2 - |\nabla u|^2\Big) \; dxdt \\
&= -\int_0^T \int_{\mathbb{R}^3} \nabla\Big(f(r)\cdot\frac{x}{r}\Big)\Big((\partial_t u)^2 - |\nabla u|^2\Big) \; dxdt \\
&= -\int_0^T \int_{\mathbb{R}^3} \Big(\frac{x}{r}f'(r)\cdot\frac{x}{r} + \frac{f(r)}{r}\cdot\nabla\cdot x - \frac{x}{r^2}\cdot f(r)\nabla r\Big)\Big((\partial_t u)^2 - |\nabla u|^2\Big) \; dxdt \\
&= -\int_0^T \int_{\mathbb{R}^3} \Big(f'(r) + \frac{2f(r)}{r}\Big)\Big((\partial_t u)^2 - |\nabla u|^2\Big) \; dxdt.
\end{aligned}
\tag{2.3}
$$

Collecting the results in (2.2) and (2.3) together, we have

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^3} \Box u f(r) \partial_r u \; dxdt \\
&= \int_{\mathbb{R}^3} \partial_t u \big(f(r) \partial_r u \big) \; dx \Big|_0^T + \int_0^T \int_{\mathbb{R}^3} f'(r) (\partial_r u)^2 \; dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} |\not\nabla u|^2 \; dxdt \\
&\quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \Big(f'(r) + \frac{2f(r)}{r} \Big) \Big((\partial_t u)^2 - |\nabla u|^2 \Big) \; dxdt. \qquad (2.4)
\end{aligned}
$$

It will be advantageous to cancel the last term, and this can be accomplished by modifying the multiplier. Another multiplier that we apply is $\frac{f(r)}{r} u$. The computation follows as

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^3} \Box u \frac{f(r)}{r} u \; dxdt \\
&= \int_0^T \int_{\mathbb{R}^3} (\partial_t^2 u - \Delta u) \frac{f(r)}{r} u \; dxdt \\
&= \int_{\mathbb{R}^3} \frac{f(r)}{r} u \partial_t u \; dx \Big|_0^T - \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} (\partial_t u)^2 \; dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \nabla u \cdot \nabla \Big(\frac{f(r)}{r} \Big) u \; dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} |\nabla u|^2 \; dxdt \\
&= \int_{\mathbb{R}^3} \frac{f(r)}{r} u \partial_t u \; dx \Big|_0^T - \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} (\partial_t u)^2 \; dxdt + \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} |\nabla u|^2 \; dxdt \\
&\quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \nabla \Big(\frac{f(r)}{r} \Big) \cdot \nabla (u)^2 \; dxdt \\
&= \int_{\mathbb{R}^3} \frac{f(r)}{r} u \partial_t u \; dx \Big|_0^T - \int_0^T \int_{\mathbb{R}^3} \frac{f(r)}{r} \Big((\partial_t u)^2 + |\nabla u|^2 \Big) \; dxdt - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \Delta \Big(\frac{f(r)}{r} \Big) u^2 \; dxdt. \qquad (2.5)
\end{aligned}
$$

Note that $\Delta(\frac{f}{r}) = \frac{1}{r^2} \partial_r (r^2 \partial_r (\frac{f}{r}))$.

Now we add the two multipliers together to form a new multiplier and obtain the following terms from (2.4) and (2.5).

$$
\begin{aligned}
0 &= \int_0^T \int_{\mathbb{R}^3} \Box u \Big(\frac{f(r)}{r} u + f(r) \partial_r u \Big) \; dxdt \\
&= \int_{\mathbb{R}^3} f(r) \partial_t u \partial_r u \; dx \Big|_0^T + \int_{\mathbb{R}^3} \frac{f(r)}{r} u \partial_t u \; dx \Big|_0^T + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f'(r) (\partial_t u)^2 \; dxdt \\
&\quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} f'(r) (\partial_r u)^2 \; dxdt + \int_0^T \int_{\mathbb{R}^3} \Big(\frac{f(r)}{r} - \frac{1}{2} f'(r) \Big) |\not\nabla u|^2 \; dxdt - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \Delta \Big(\frac{f(r)}{r} \Big) u^2 dxdt. \qquad (2.6)
\end{aligned}
$$

In order to obtain a meaningful estimate, the function f here has to satisfy the following properties to guarantee that the last four terms above are positive.

- $f \in C^2$
- f bounded

- $f > 0$
- $f' > 0$
- $\frac{f(r)}{r} - f' > 0$
- $-\Delta(\frac{f}{r}) > 0.$

Inspired by the inverse tangent function, we find $f(r) = \frac{r}{r+R}, R > 0$ to be a good candidate; it satisfies all the requirements. Here we check the last three.

- $f'(r) = \frac{R}{(r+R)^2} > 0$
- $\frac{f(r)}{r} - f'(r) = \frac{1}{r+R} - \frac{R}{(r+R)^2} = \frac{r}{(r+R)^2} > 0$
- $-\Delta(\frac{f(r)}{r}) = -r^2\partial_r(r^2\partial_r(\frac{1}{r+R})) = \frac{2R}{r(r+R)^3} > 0.$

Plugging f back in (2.6), we get

$$\begin{aligned}
0 = &\int_{\mathbb{R}^3} \frac{r}{r+R}\partial_t u \partial_r u \; dx\Big|_0^T + \int_{\mathbb{R}^3} \frac{1}{r+R} u\partial_t u \; dx\Big|_0^T + \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \frac{R}{(r+R)^2}(\partial_t u)^2 \; dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \frac{R}{(r+R)^2}(\partial_r u)^2 \; dxdt + \int_0^T \int_{\mathbb{R}^3} \Big(\frac{1}{r+R} - \frac{1}{2}\frac{R}{(r+R)^2}\Big)|\nabla\!\!\!/ u|^2 \; dxdt \\
&+ \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \frac{2R}{r(r+R)^3} u^2 dxdt.
\end{aligned}$$

Inside the compact set where $|x| \leq R$, we consider the last four terms first, and we could see that

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \frac{R}{(r+R)^2}(\partial_t u)^2 \; dxdt + \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \frac{R}{(r+R)^2}(\partial_r u)^2 \; dxdt \\
&+ \int_0^T \int_{\mathbb{R}^3} \Big(\frac{1}{r+R} - \frac{1}{2}\frac{R}{(r+R)^2}\Big)|\nabla\!\!\!/ u|^2 \; dxdt + \frac{1}{2}\int_0^T \int_{\mathbb{R}^3} \frac{2R}{r(r+R)^3} u^2 dxdt \\
\gtrsim &R^{-1}\int_0^T \int_{|x|\leq R} \Big((\partial_t u)^2 + (\partial_r u)^2 + |\nabla\!\!\!/ u|^2\Big) \; dxdt + R^{-3}\int_0^T \int_{|x|\leq R} u^2 \; dxdt \\
= &R^{-1}\int_0^T \int_{|x|\leq R} \Big((\partial_t u)^2 + (\nabla u)^2\Big) \; dxdt + R^{-3}\int_0^T \int_{|x|\leq R} u^2 \; dxdt.
\end{aligned}$$

We are only left to show that

$$\int_{\mathbb{R}^3} f(r)\partial_t u \partial_r u \; dx\Big|_0^T + \int_{\mathbb{R}^3} \frac{f(r)}{r} u\partial_t u \; dx\Big|_0^T \lesssim E[u](0). \tag{2.7}$$

This can be achieved by applying the Cauchy-Schwarz inequality and the Hardy inequality to the two terms on the left hand side respectively. The procedures here are very similar to what was done with the Morawetz estimates.

Thus, we have established that, for all $R > 0$,

$$R^{-1}||u'||^2_{L^2_t L^2_x([0,T]\times|x|\leq R)} + R^{-3}||u||^2_{L^2_t L^2_x([0,T]\times|x|\leq R)} \lesssim ||u'(0,\cdot)||^2_{L^2}. \tag{2.8}$$

Since here the right hand side is independent of R, taking the supremum of the left hand side of (2.8) yields the desired estimate. $\square$

Chapter 3

Local Energy Estimates with Degenerate Trappings

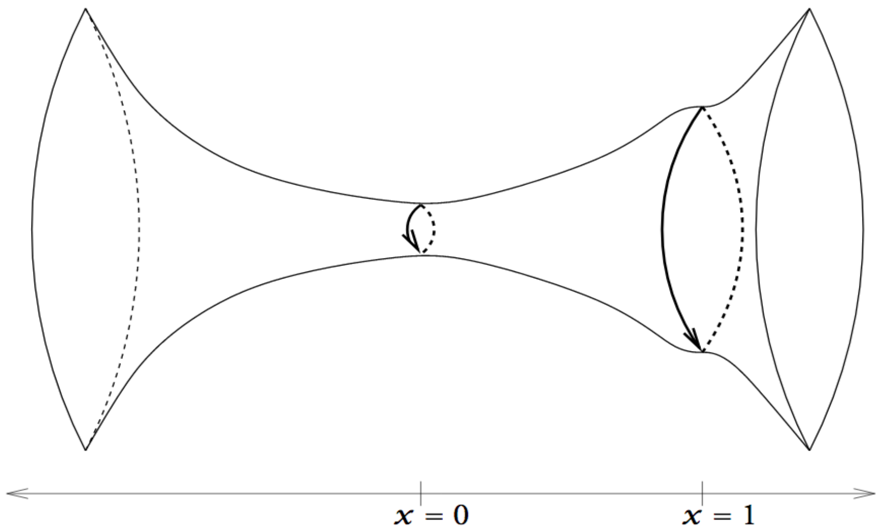
We now study the wave equation on a warped product manifold whose generating function has an inflection point which corresponds to a surface of trapped null geodesics.

We begin by describing the geometry. We consider the manifold with $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$ equipped with the Lorentzian metric

$$-ds^2 = -dt^2 + dx^2 + A(x)^2 d\sigma_{\mathbb{S}^2}^2.$$

Let m_1, and m_2 be positive integers. The generating function of the surface of revolution is defined piecewise by

$$A(x) = \begin{cases} \left(1 + \int_0^x a(y)\ dy\right)^{1/2} \text{where} \quad a(x) = \dfrac{x^{2m_1 - 1}(x-1)^{2m_2}}{(1+x^2)^{m_1+m_2-1}} & \text{for} \quad x \geq 0, \\ (x^{2m_1} + 1)^{\frac{1}{2m_1}} & \text{for} \quad x < 0. \end{cases}$$



[1]

For $x \geq 0$, there exists an inflection point at $x = 1$, where the trapping is said to be semistable, that is, stable from the right direction and unstable from the left direction. Also notice that $a(x) \approx |x|$ and $A^2(x) \approx x^2$ as $|x|$ goes to infinity, so the surface is asymptotically Minkowski and has two trapped sets.

Note that unstable trapping occurs at $x = 0$, and that for $x < 0$, when $|x| \to \infty$, the surface becomes asymptotically flat and can be approximated by $|x|$.

We use $\Box_{\mathfrak{g}} u = \partial_t^2 u - \frac{1}{A(x)^2}\partial_x(A(x)^2\partial_x u) - \not\nabla \cdot \not\nabla u$. Note that here we represent the calculation in polar coordinate, where $dV = A(x)^2 dx d\sigma(\omega) dt$. Here x is one dimensional. $\not\nabla_0 u$ denotes the angular derivative of u that is tangential to the unit sphere, and $\not\nabla u = \frac{1}{A(x)}\not\nabla_0 u$.

As this metric is static, there is a natural coercive energy, which is conserved when $\Box u = 0$

$$E[u](t) = \int (\partial_t u)^2 + (\partial_x u)^2 + \frac{1}{A(x)^2}|\not\nabla_0 u|^2 \; dV = \int |\partial u|^2 \; dV.$$

Here we use $\partial u = (\partial_t u, \partial_x u, \not\nabla_0 u)$, so we also have $|\partial u|^2 = (\partial_t u)^2 + (\partial_x u)^2 + \frac{1}{A(x)^2}|\not\nabla_0 u|^2$.

The conservation of energy can be proved by multiplying $\Box_{\mathfrak{g}} u$ by $\partial_t u$ and integrating by parts. More generally, it yields

$$E[u](t) \lesssim E[u](0) + \Big| \int_0^T \Box_{\mathfrak{g}} u \partial_t u \; dV dt \Big|. \tag{3.1}$$

The LE norm that we shall use in describing the estimates is defined to be

$$\|u\|_{LE} = \sup 2^{-j/2} \|u\|_{L^2 L^2([0,T]\times \langle x \rangle \approx 2^j)},$$

[1]The figure is from [15].

and we also use LE^1 norm which is defined to be

$$\|u\|_{LE^1} = \Big\|(\partial u, \langle x\rangle^{-1}u)\Big\|_{LE}.$$

We shall use L^2L^2 to indicate the the full space-time L^2 norm to mimic what is commonly seen on Minkowski space, with the first L^2 being in t and the second L^2 being over the spatial variables. See, e.g. [13]. The norms in t will be taken over [0,T], like what was seen in the previous chapters. All constants will be independent of T, which yields the desired global estimates. The notation LE_R is used to indicate the LE norm restricted to a annulus with $2^j \approx R$, and similarly, $LE_{>R}$ indicates the LE norm with the restriction that $2^j > R$.

Using the product structure of the metric, we will separate space and time in the volume form and indicate $dV = A(x)^2\ dx d\sigma_{\mathbb{S}^2}$. We will use $dV\ dt$ throughout the rest of the paper since the volume form of the full space-time is desired.

A more generalized calculation using the multiplier method and integration by parts is shown below, and this provides the foundation of all our future estimates on this specific geometry. Here in the calculation we have $f(x)$ and $g(x)$ to be generic multipliers first and later in this chapter we will specify them in deriving our estimates.

Suppose w, $g \in C^2$ and $f \in C^1$, and and for each t, solution $w(t,x)$ vanishes for large enough $|x|$. Then

$$\begin{aligned}
&-\int_0^T\int \Box_{\mathfrak{g}} w\big\{f(x)\partial_x w + g(x)w\big\}\ dV dt \\
&= \int \partial_t w f(x)\partial_x w\ dV\Big|_0^T + \int_0^T\int f'(x)(\partial_x w)^2\ dV dt + \int_0^T\int \frac{f(x)A'(x)}{A(x)}\cdot\frac{1}{A(x)^2}|\nabla\!\!\!/_0 w|^2\ dV\ dt \\
&\quad -\frac{1}{2}\int_0^T\int\Big(f'(x) + 2\frac{f(x)}{A(x)}\Big)\Big((\partial_t w)^2 - (\partial_x w)^2 - \frac{1}{A(x)^2}\cdot|\nabla\!\!\!/_0 w|^2\Big)\ dV\ dt + \int \partial_t w g(x) w\ dV\Big|_0^T \\
&\quad -\int_0^T\int g(x)(\partial_t w)^2 dV\ dt + \int_0^T\int g(x)\big((\partial_x w)^2 + \frac{1}{A(x)^2}\cdot|\nabla\!\!\!/_0 w|^2\big)dV\ dt \\
&= \int \partial_t w(f(x)\partial_x w + g(x)w)\ dV\ \Big|_0^T - \frac{1}{2}\int_0^T\int\frac{1}{A(x)^2}\partial_x(A(x)^2\partial_x g)w^2\ dV\ dt \\
&\quad + \int_0^T\int\Big\{f'(x) + g(x) - \frac{1}{2}\Big(A(x)^{-2}\partial_x(A(x)^2 f(x))\Big)\Big\}(\partial_x w)^2\ dV\ dt \\
&\quad + \int_0^T\int\Big\{f(x)\frac{A'(x)}{A(x)} + g(x) - \frac{1}{2}\Big(A(x)^{-2}\partial_x(A(x)^2 f(x))\Big)\Big\}\frac{1}{A(x)^2}|\nabla\!\!\!/_0 w|^2\ dV dt \\
&\quad + \int_0^T\int\Big\{-g(x) + \frac{1}{2}\Big(A(x)^{-2}\partial_x(A(x)^2 f(x))\Big)\Big\}(\partial_t w)^2\ dV dt.
\end{aligned} \tag{3.2}$$

3.1 Exterior Estimate

In this section, we want to establish an exterior estimate when the solution is cut off away from $x = 0$ and $x = 1$. This estimate shows that the local energy estimates necessarily hold near the infinite ends with a lower order error term that is supported on a compact region.

Proposition 3.1.1. *For any parameters R and R_1 that satisfy $\frac{1}{2}R \geq R_1$ with R_1 being sufficiently large, we have*

$$\|u\|^2_{LE^1{}_{|x|>R}} \lesssim E[u](0) + \int_0^T \int |\Box_{\mathfrak{g}} u| \Big(|\partial u| + \frac{1}{A(x)}|u|\Big)\; dV dt + \frac{1}{R}\frac{1}{R_1}\|u\|^2_{LE_{|x|\approx R}}.$$

Proof. We first consider the case when $x < 0$. This part of the proof is a direct analog of [13, Proposition 2.3].

In order to attain a meaningful estimate, we have

$$f(x) = (1 - \beta(|x|/R_1))h(x), \qquad\qquad h(x) = \frac{x}{|x| + \rho}, \rho \geq R,$$
$$g(x) = \frac{1}{2}A(x)^{-2}h(x)\partial_x\big[(1 - \beta(|x|/R_1))A(x)^2\big]$$

to be applied to the result from the integration in (3.2). Here we define $\beta(\rho)$ as a smooth, monotonically decreasing cutoff that is 1 when $\rho < \frac{1}{2}$ and 0 when $\rho > 1$. We compute

$$h'(x) = \frac{\rho}{(|x| + \rho)^2}, \qquad\qquad h''(x) = -\frac{2\rho \cdot \operatorname{sgn}(x)}{(|x| + \rho)^3}.$$

Starting from the coefficient of the time derivative, we see that

$$\begin{aligned}
&- g(x) + \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big(A(x)^2 f(x)\big)\Big\}\\
=& -\frac{1}{2}A(x)^{-2}h(x)\partial_x\big((1 - \beta(|x|/R_1))A(x)^2\big) + \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big(A(x)^2(1 - \beta(|x|/R_1))h(x)\big)\Big\}\\
=& -\frac{1}{2}A(x)^{-2}h(x)\partial_x\big((1 - \beta(|x|/R_1))A(x)^2\big) + \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big(A(x)^2(1 - \beta(|x|/R_1)\big)h(x)\Big\}\\
&+ \frac{1}{2}\Big\{A(x)^{-2}\big(A(x)^2(1 - \beta(|x|/R_1)\big)h'(x)\Big\}\\
=&\frac{1}{2}\big((1 - \beta(|x|/R_1)\big)\frac{\rho}{(|x| + \rho)^2}.
\end{aligned} \tag{3.3}$$

It can be easily seen that this is everywhere non-negative. Moreover, this can be bounded below by $1/\rho$ when $|x| \approx \rho$.

Using (3.3), for the radial derivatives, it follows that,

$$\begin{aligned}
&f'(x) + g(x) - \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big(A(x)^2 f(x)\big)\Big\}\\
=&\big(1 - \beta(|x|/R_1)\big)h'(x) - \frac{1}{R_1}\beta'\big(|x|/R_1\big)\operatorname{sgn}(x)h(x) - \frac{1}{2}\big((1 - \beta(|x|/R_1)\big)h'(x)\\
=& -\frac{1}{R_1}\beta'(|x|/R_1)\operatorname{sgn}(x)\frac{x}{|x| + \rho} + \frac{1}{2}\big((1 - \beta(|x|/R_1)\big)\frac{\rho}{(|x| + \rho)^2}.
\end{aligned} \tag{3.4}$$

Since β is monotonically decreasing, the first term above is non-negative. Thus, this coefficient is bounded below by $\frac{1}{2}\big((1-\beta(|x|/R_1)\big)\frac{\rho}{(|x|+\rho)^2}$.

For the angular derivatives, we have

$$\begin{aligned} f(x)\frac{A'(x)}{A'(x)}+g(x)-\frac{1}{2}\Big\{A(x)^{-2}\partial_x\big(A(x)^2f(x)\big)\Big\} =&(1-\beta(|x|/R_1))h(x)\frac{A'(x)}{A'(x)}-\frac{1}{2}\big((1-\beta(|x|/R_1)\big)h'(x)\\ =&(1-\beta(|x|/R_1)\Big(\frac{A'(x)}{A(x)}h(x)-\frac{1}{2}h'(x)\Big). \end{aligned}$$

We further examine the second factor. Here we have

$$\frac{A'(x)}{A(x)}h(x)-\frac{1}{2}h'(x)=\frac{x^{2m}}{x^{2m}+1}\cdot\frac{1}{|x|+\rho}-\frac{1}{2}\frac{\rho}{(|x|+\rho)^2}. \tag{3.5}$$

Since $(1-\beta(|x|/R_1))$ is supported on $|x|>R_1$, we have $\frac{x^{2m}}{x^{2m}+1}\geq\frac{1}{2}$ for R_1 sufficiently large, and thus the quantity in (3.5) is bounded below by $\frac{|x|}{2(|x|+\rho)^2}$. When $|x|\approx\rho$ that is sufficiently large, the entire coefficient can be approximated from below by $1/\rho$.

Direct computation and then factorization of the lower order term yield

$$\begin{aligned} &-\frac{1}{4}\Big\{A(x)^{-2}\partial_x\Big[A(x)^2\partial_x\Big(h(x)A(x)^{-2}\partial_x(A(x)^2)\Big)\Big]\Big\}\\ &\quad=-\frac{1}{2}\frac{x^{2m-1}}{1+x^{2m}}h''(x)+\frac{2m-1}{(1+x^{2m})^2}x^{2m-2}\Big(\frac{h(x)}{x}-h'(x)\Big)+\frac{2m-1}{(1+2m)^3}x^{2m-3}m(x^{2m}-1)h(x). \end{aligned}$$

Here the first term is everywhere non-negative and is $\gtrsim\rho^{-3}$ when $|x|\approx\rho$. The second term is also non-negative since $\frac{h(x)}{x}-h'(x)\geq 0$. Finally, the last term is easily seen to be non-negative on the support of $(1-\beta(|x|/R_1))$.

We also need to account for the error term that results from when derivatives land on the cut-off. Here we see that

$$\begin{aligned} -\frac{1}{2}A(x)^{-2}\partial_x[A(x)^2\partial_xg(x)]+\frac{1}{4}((1-\beta(|x|/R_1))\Big\{A(x)^{-2}\partial_x\Big[A(x)^2\partial_x\Big(h(x)A(x)^{-2}\partial_x(A(x)^2)\Big)\Big]\Big\}\\ \lesssim\frac{1}{\rho}\frac{1}{R_1^2}\mathbf{1}_{|x|\approx R_1}. \end{aligned} \tag{3.6}$$

Now we consider the case where $x\geq 0$. When we examine the coefficients, as we did for the previous case, our estimates for the time derivatives and the radial derivatives work exactly the same, as we did not use any specific properties of $A(x)$. It is the angular derivatives that require additional care when we try to determine a lower bound. Here we have

$$\frac{A'(x)}{A(x)}h(x)-\frac{1}{2}h'(x)=\frac{a}{2A^2}h(x)-\frac{1}{2}h'(x)\geq\frac{|x|}{2x^2}h(x)-\frac{1}{2}h'(x)=\frac{x}{2(|x|+\rho)^2}. \tag{3.7}$$

To establish the above inequality, it suffices to show that, for sufficiently large x,

$$\frac{a}{A^2}-\frac{1}{x}\geq 0. \tag{3.8}$$

To prove such, we multiply by $\partial_y y$ and integrate by parts to see that

$$ax - A^2 = ax - \Big(1 + \int_0^x a(y)\partial_y y \; dy\Big) = ax - 1 - ax + \int_0^x a'(y)y \; dy.$$

We can write the integrand $a'(y)y$ explicitly, which is

$$a'(y)y = (y-1)^{4m_2-1}y^{4m_1-3}\left(y^2+1\right)^{-2m_1-2m_2+1}\left(2m_1(y-1)+(y+1)\left(2m_2y+(y-1)^2\right)\right)$$

and there exist a y_0 sufficiently large so that $a'(y_0)y_0 > 1$. Thus we have, for sufficiently large x,

$$\int_0^x a'(y)y \; dy > 1,$$

proving the inequality. Moreover, the right hand side of (3.7) is $\approx \frac{1}{\rho}$ for sufficiently large ρ.

Here we record the results of the two other coefficients. For the time derivatives, we have

$$\begin{aligned}
&- g(x) + \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big[A(x)^2 f(x)\big]\Big\} \\
=& - \frac{1}{2}A(x)^{-2}h(x)\partial_x\big[(1-\beta(|x|/R_1))A(x)^2\big] + \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big[A(x)^2(1-\beta(|x|/R_1))h(x)\big]\Big\} \\
=& - \frac{1}{2}A(x)^{-2}h(x)\partial_x\big[(1-\beta(|x|/R_1))A(x)^2\big] + \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big[A(x)^2(1-\beta(|x|/R_1)\big]h(x)\Big\} \\
&+ \frac{1}{2}\Big\{A(x)^{-2}\big[A(x)^2(1-\beta(|x|/R_1)\big]h'(x)\Big\} \\
=&\frac{1}{2}\big[(1-\beta(|x|/R_1)\big]h'(x)\big].
\end{aligned}$$

which can be approximated from below by $\frac{1}{\rho}$ at $x \approx \rho$.

For the coefficient of the ∂_x term, it simply follows, with the result of the time derivative, that

$$\begin{aligned}
&f'(x) + g(x) - \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big[A(x)^2 f(x)\big]\Big\} \\
=&(1-\beta(|x|/R_1))h'(x) - \frac{1}{R_1}\beta'(|x|/R_1)\mathrm{sgn}(x)h(x) - \frac{1}{2}\big[(1-\beta(|x|/R_1)\big]h'(x) \\
=& - \frac{1}{R_1}\beta'(|x|/R_1)\frac{|x|}{|x|+\rho} + \frac{1}{2}\big[(1-\beta(|x|/R_1)\big]h'(x).
\end{aligned}$$

Since β is monotonically decreasing, the first term above is non-negative. Thus the coefficient is bounded below by $\frac{1}{2}\big[(1-\beta(|x|/R_1)\big]\frac{\rho}{(|x|+\rho)^2}$, which can also be approximated from below by $\frac{1}{\rho}$ when $|x| \approx \rho \geq R$.

Now we compute the coefficient of the lower order term.

$$\begin{aligned}
&- \frac{1}{4}\Big\{A(x)^{-2}\partial_x\Big[A(x)^2\partial_x\Big(h(x)A(x)^{-2}\partial_x(A(x)^2)\Big)\Big]\Big\} \\
=& - \frac{a(x)h(x)(A'(x))^2}{2A(x)^4} + \frac{A'(x)h(x)a'(x)}{2A(x)^3} + \frac{a(x)A'(x)h'(x)}{2A(x)^3} - \frac{a'(x)h'(x)}{2A(x)^2} + \frac{a(x)h(x)A''(x)}{2A(x)^3} \\
&- \frac{h(x)a''(x)+a(x)h''(x)}{4A(x)^2}.
\end{aligned}$$

Meanwhile, we make the observation that

$$A'(x) = \frac{1}{2}\Big(1+\int_0^x a(y)\ dy\Big)^{-\frac{1}{2}} \cdot a(x) = \frac{a}{2A}, \qquad\qquad A''(x) = -\frac{a^2}{4A^3} + \frac{a'}{2A}.$$

Thus the coefficient of the lower order term can be further simplified to

$$-\frac{a(x)^3 h(x)}{4A(x)^6} + \frac{a(x)h(x)a'(x)}{2A(x)^4} + \frac{a(x)^2 h'(x)}{4A(x)^4} - \frac{a'(x)h'(x)}{2A(x)^2} - \frac{h(x)a''(x) + a(x)h''(x)}{4A(x)^2}.$$

Here we also compute

$$a'(x) = (x-1)^{2m_2-1}x^{2m_1-2}\left(x^2+1\right)^{-m_1-m_2}\left(2m_1(x-1) + (x+1)\left(2m_2 x + (x-1)^2\right)\right),$$

and

$$\begin{aligned} a''(x) =& 2(x-1)^{2m_2-2}x^{2m_1-3}\left(x^2+1\right)^{-m_1-m_2-1} \\ & \Big(2m_2^2x^2(x+1)^2 - m_1\left((x-1)^2\left(x^2+3\right) - 4m_2 x\left(x^2-1\right)\right) - 2m_2x\left(x^3+x^2+x-1\right) \\ & + 2m_1^2(x-1)^2 + \left(x^2+1\right)(x-1)^2\Big). \end{aligned}$$

Writing the derivatives in terms of $a(x)$, we obtain the following alternate representations that might be proved to be more useful.

$$\begin{aligned} a'(x) &= \frac{(2m_1-1)}{x}a(x) + \frac{2m_2}{x-1}a(x) - \frac{m_1+m_2-1}{1+x^2}2xa(x) \\ &= a(x)\Big(\frac{2m_1-1}{x} + \frac{2m_2}{x-1} - \frac{m_1+m_2-1}{1+x^2}2x\Big), \end{aligned}$$

and

$$\begin{aligned} a''(x) =& a'(x)\Big(\frac{2m_1-1}{x} + \frac{2m_2}{x-1} - \frac{m_1+m_2-1}{1+x^2}2x\Big) \\ & + a(x)\Big(-\frac{2m_1-1}{x^2} - \frac{2m_2}{(x-1)^2} + \frac{m_1+m_2-1}{(1+x^2)^2}4x^2 - \frac{2(m_1+m_2-1)}{1+x^2}\Big) \\ =& a(x)\Big(\frac{2m_1-1}{x} + \frac{2m_2}{x-1} - \frac{m_1+m_2-1}{1+x^2}2x\Big)^2 \\ & + a(x)\Big(-\frac{2m_1-1}{x^2} - \frac{2m_2}{(x-1)^2} + \frac{m_1+m_2-1}{(1+x^2)^2}4x^2 - \frac{2(m_1+m_2-1)}{1+x^2}\Big). \end{aligned} \tag{3.9}$$

Using these two results and re-grouping the terms in the coefficient, we get

$$\begin{aligned} & h''(x)\Big(-\frac{a(x)}{4A(x)^2}\Big) + h(x)\Big(-\frac{a(x)^3}{4A(x)^6} + \frac{a(x)a'(x)}{2A(x)^4} - \frac{a''(x)}{4A(x)^2} + \frac{a(x)^2}{x\cdot 4A(x)^2} - \frac{a'(x)}{x\cdot 2A(x)^2}\Big) \\ & + \Big(\frac{h(x)}{x} - h'(x)\Big)\Big(-\frac{a(x)^2}{4A(x)^4} + \frac{a'(x)}{2A(x)^2}\Big) \\ =& h''(x)\Big(-\frac{a(x)}{4A(x)^2}\Big) + h(x)\Big(\Big(\frac{a}{A^2} - \frac{1}{x}\Big)\Big(-\frac{a(x)^2}{4A(x)^4} + \frac{a'(x)}{2A(x)^2}\Big) - \frac{a''(x)}{4A(x)^2}\Big) \\ & + \Big(\frac{h(x)}{x} - h'(x)\Big)\Big(-\frac{a(x)^2}{4A(x)^4} + \frac{a'(x)}{2A(x)^2}\Big). \end{aligned} \tag{3.10}$$

Now we are ready to analyze each of the three terms in (3.10). The first term is everywhere non-negative and is $\geq \rho^{-3}$ when $|x| \approx \rho$. $\left(\frac{h(x)}{x} - h'(x)\right)$ is easily seen to be everywhere non-negative, and since we have already established that $\frac{a(x)}{A(x)^2} \gtrsim \frac{|x|}{x^2} = \frac{1}{|x|}$, the other factor in the third term becomes

$$\frac{a(x)}{A(x)^2}\Big(-\frac{a(x)}{4A(x)^2}\Big) + \frac{a'(x)}{2A(x)^2} \gtrsim -\frac{1}{|x|}\frac{a(x)}{4A(x)^2} + \frac{a'(x)}{2A(x)^2}. \tag{3.11}$$

To see that the right hand side of (3.11) is non-negative, when x is sufficiently large, direct algebraic computation shows that

$$\frac{2m_1 - 1}{x} + \frac{2m_2}{x-1} - \frac{m_1 + m_2 - 1}{1 + x^2}2x - \frac{1}{2x} \geq 0.$$

Also, for sufficiently large x, $h''(x)\big(-\frac{a(x)}{4A(x)^2}\big)$, which is bounded from below by x^{-4}, will dominate over $-\frac{a''(x)}{4A(x)^2}$, which decays at the rate of $-x^{-6}$. If we combine the factors of $a(x)$ in (3.9) and simplify, it will become clear that $a''(x)$ decays like x^{-4}.

Thus, the coefficient of the lower order term is $\gtrsim \rho^{-3}$ when $|x| \approx \rho$.

We also need to account for the error term that results from when derivatives land on the cut-off with $|x| \approx R_1$. We see that

$$\begin{aligned}-\frac{1}{2}A(x)^{-2}\partial_x\big(A(x)^2\partial_x g(x)\big) + \frac{1}{4}((1-\beta(|x|/R_1))\Big\{A(x)^{-2}\partial_x\Big[A(x)^2\partial_x\Big(h(x)A(x)^{-2}\partial_x(A(x)^2)\Big)\Big]\Big\} \\ \lesssim \frac{1}{\rho}\frac{1}{R_1^2}\mathbf{1}_{|x|\approx R_1}.\end{aligned}$$

We have established estimates for each of the coefficient in (3.2) and the error term for $x \geq 0$ and $x < 0$ separately. Now we can consider the entire domain. Substituting each of the estimates into (3.2), we have

$$\begin{aligned}-\int_0^T\int \Box_{\mathfrak{g}} u\big\{f(x)\partial_x u + g(x)u\big\}\ dV dt - \int \partial_t u(f(x)\partial_x u + g(x)u)\ dV\Big|_0^T \\ \gtrsim \frac{1}{\rho}\int_0^T\int_{|x|\approx\rho}|\partial u|^2\ dV dt + \frac{1}{\rho^3}\int_0^T\int_{|x|\approx\rho}u^2\ dV dt - \frac{1}{R_1^2\rho}\int_0^T\int_{|x|\approx R_1}u^2 dV dt.\end{aligned} \tag{3.12}$$

The reason why we take a negative sign of the error term is that this provides the worse case scenario since we are trying to establish a lower bound.

We first focus on the time boundary term in (3.12). The Cauchy-Schwarz inequality and the fact that f is bounded give

$$\int f(x)\partial_t u\partial_x u\ dV\Big|_0^T \lesssim E[u](t) + E[u](0) \lesssim E[u](0) + \Big|\int_0^T \Box_{\mathfrak{g}} u\partial_t u\ dV dt\Big|.$$

For the other term,

$$\int \partial_t u g(x) u\ dV = \frac{1}{2}\int A(x)^{-2}h(x)\partial_x\Big((1-\beta(|x|/R_1))A(x)^2\Big)u\partial_t u\ dV,$$

we first compute

$$A(x)^{-2}h(x)\partial_x\Big(\beta(|x|)A(x)^2\Big) = h(x)\beta'(|x|) + \beta(|x|)h(x)\frac{a(x)}{A(x)^2}.$$

Since β is monotonically decreasing and $\beta(|x|)$ and $h(x)$ are both bounded above by 1, we have

$$A(x)^{-2}h(x)\partial_x\Big(\beta(|x|)A(x)^2\Big) \lesssim \frac{1}{A(x)}.$$

Now recalling that $dV = A(x)^2 dx\ d\sigma_{\mathbb{S}^2}$ and integrating by parts, we see that

$$\int \frac{u^2}{A(x)^2}\ dV = \int \frac{u^2}{A(x)^2}A(x)^2\ dx\ d\sigma = \int u^2\partial_x x\ dx\ d\sigma = -2\int x\ u\partial_x u\ dx\ d\sigma.$$

Applying the Cauchy-Schwarz inequality , it follows that

$$\int \frac{u^2}{A(x)^2}\ dV \lesssim \int \Big|\frac{x}{A(x)}\Big|\Big|\frac{u}{A(x)}\Big||\partial_x u|\ dV \lesssim \Big(\int A(x)^{-2}u^2\ dV\Big)^{1/2}\Big(\int (\partial_x u)^2\ dV\Big)^{1/2}.$$

Thus, we obtain

$$\Big(\int \frac{u^2}{A(x)^2}\ dV\Big)^{1/2} \lesssim \Big(\int (\partial_x u)^2\ dV\Big)^{1/2}.$$

which is a variant of a Hardy inequality.

Applying these bounds to the time boundary terms in (3.12), we have

$$\begin{aligned}\sup_{t\in[0,T]} E[u](t) + \rho^{-1}\int_0^T\int |\partial u|^2\ dV dt + \rho^{-3}\int_0^T\int_{|x|\approx\rho} u^2\ dV dt \\ \lesssim E[u](0) + \int_0^T\int |\Box_{\mathfrak{g}} u|\Big(|\partial u| + \frac{|u|}{A(x)}\Big)\ dV dt + \rho^{-1}R_1^{-1}\|u\|^2_{LE_{R_1}}.\end{aligned} \tag{3.13}$$

$\square$

3.2 Low Frequency Estimate

We now establish a local energy estimate for sufficiently low time-frequencies.

Lemma 3.2.1. *For any $R > 0$, we have*

$$\begin{aligned}\Big\|A(x)^{-1/2}\partial_x u\Big\|^2_{L^2L^2_{|x|<R/2}} &+ \Big\|A(x)^{-3/2}\nabla\!\!\!/_0 u\Big\|^2_{L^2L^2_{|x|<R/2}} + \Big\|A(x)^{-1/2}\partial_t u\Big\|^2_{L^2L^2_{|x|<R/2}} \\ &+ \Big\|\langle x\rangle^{-3/2}u\Big\|^2_{L^2L^2_{|x|<R/2}} \\ &\lesssim E[u](0) + \Big\|A(x)^{-1/2}\partial_t u\Big\|^2_{L^2L^2_{|x|<R}} \\ &+ \int_0^T\int |\Box_{\mathfrak{g}} u|\Big(|\partial_t u| + \frac{1}{A(x)}|u|\Big)\ dV dt + \|u\|^2_{LE^1_{|x|\approx R}}.\end{aligned}$$

Proof. Here we apply (3.2) with the multipliers $f \equiv 0$ and $g(x) = \frac{1}{A(x)}$. This gives

$$
\begin{aligned}
-\int_0^T \int \Box_{\mathfrak{g}} w \frac{1}{A(x)} w \; dV dt = \int \partial_t w \frac{1}{A(x)} w \; dV \Big|_0^T - \frac{1}{2} \int_0^T \int \frac{1}{A(x)^2} \partial_x (A(x)^2 \partial_x g) w^2 \; dV dt \\
+ \int_0^T \int \frac{1}{A(x)} (\partial_x w)^2 \; dV dt + \int_0^T \int \frac{1}{A(x)^3} |\nabla\!\!\!/_0 w|^2 \; dV dt - \int_0^T \int \frac{1}{A(x)} (\partial_t w)^2 \; dV dt.
\end{aligned}
$$

Rearranging, we have

$$
\begin{aligned}
\int_0^T \int \frac{1}{A(x)} \Big[(\partial_x w)^2 + \frac{1}{A(x)^2} |\nabla\!\!\!/_0 w|^2 \Big] \; dV dt + \frac{1}{2} \int_0^T \int \Big(- \frac{a'}{2A^3} + \frac{a^2}{4A^5} \Big) w^2 \; dV dt \\
= - \int \frac{1}{A(x)} w \partial_t w \; dV \Big|_0^T - \int_0^T \int \Box_{\mathfrak{g}} w \frac{1}{A(x)} w \; dV dt + \int_0^T \int \frac{1}{A(x)} (\partial_t w)^2 \; dV dt.^{2}
\end{aligned}
\tag{3.14}
$$

The vanishing coefficients are the "loss" that is due to the trapping. This loss is only necessary for the directions tangent to the trapping (angular and time in this case). Experience in other situations indicates that the loss is not typically needed in the lower order term, see, e.g. [13]. We now show that the vanishing in the coefficient at the origin and at $x = 1$ of the second term on the left side of (3.14) can be eliminated.

First, consider $\beta(|x|/\epsilon)$. ϵ here is chosen so that $\beta(|x|)/\epsilon)$ is supported on a neighborhood around $x = 0$ that does not contain $x = 1$. Integrating by parts, we have

$$
\begin{aligned}
&\int \beta(|x|/\epsilon) w^2 A(x)^2 dx \\
&= - \int \beta'(|x|/\epsilon)|x| w^2 A(x)^2 dx - 2 \int \beta(|x|/\epsilon) x w \partial_x w A(x)^2 dx - 2 \int \beta(|x|/\epsilon) \frac{x A'(x)}{A(x)} w^2 A(x)^2 dx.
\end{aligned}
\tag{3.15}
$$

Applying the Cauchy-Schwarz inequality to the second term and using Young's inequality with exponent 2 gives [3]

$$
\begin{aligned}
2 \int \beta(|x|/\epsilon) x w \partial_x w A(x)^2 dx &\le 2 \Big(\int \beta(|x|/\epsilon) A(x)^4 w^2 \; dx \Big)^{1/2} \Big(\int \beta(|x|/\epsilon) (\partial_x w)^2 x^2 \; dx \Big)^{1/2} \\
&\le \int \beta(|x|/\epsilon) A(x)^2 w^2 \; dx + \int \beta(|x|/\epsilon) (\partial_x w)^2 x^2 \; dx.
\end{aligned}
\tag{3.16}
$$

And, with bootstrapping, we have

$$
\int \beta(|x|/\epsilon) w^2 dV \lesssim \int \beta'(|x|/\epsilon)|x| w^2 dV + \int \beta(|x|/\epsilon) w^2 (\partial_x w)^2 dV + \int \beta(|x|/\epsilon) x \frac{a(x)}{2A(x)^2} w^2 dV.
$$

Then consider $\beta((|x-1|)/\epsilon)$. Note that here $\beta((|x-1|)/\epsilon)$ is supported on a neighborhood of $x = 1$ that does not contain $x = 0$. We can run the same argument and remove the vanishing at $x = 1$.

---

[2] At the final stage of writing this paper, it was noticed that the coefficient of the second term on the left hand side may not be everywhere nonnegative. This will fixed in future work.

[3] $ab \le \frac{a^2}{2} + \frac{b^2}{2}$

Applying the Cauchy-Schwarz inequality, we obtain

$$\int_0^T \int \frac{1}{A(x)}\Big((\partial_x w)^2 + \frac{1}{a(x)^2}|\nabla\!\!\!/_0 w|^2\Big)\, dV dt + \int_0^T \int \langle x\rangle^{-3} w^2 \, dV dt$$
$$\lesssim E[w](0) + \int_0^T \int |\Box_{\mathfrak{g}} w|\Big(|\partial_t w| + \frac{1}{A(x)}|w|\Big)\, dV dt + \int_0^T \int \frac{1}{A(x)}(\partial_t w)^2 \, dV dt.$$

Finally, we let $w = \beta(|x|/R)u$. We also need to account for the nonhomogeneous term that results from when the derivative lands on β. This can be controlled by $\|u\|^2_{LE^1}$ by computation of the commutator operation

$$\int_0^T \int [\Box_{\mathfrak{g}}, \beta(|x|/R)]u\Big(|\partial w| + \langle A(x)\rangle^{-1} w\Big) dV \; dt \lesssim \|u\|^2_{LE^1_R}, \tag{3.17}$$

which concludes the proof. □

3.3 High Frequency Estimate

In this section we establish an local energy estimate for the sufficiently high time-frequency regime, which will be supplemented by the previous exterior estimate.

Lemma 3.3.1. *For $R > 0$ sufficiently large, we have*

$$\begin{aligned}
&\frac{1}{2R}\Big(1 - \frac{\delta}{R^{4m_1}}\Big)\Big\|\Big(\frac{a(x)A(x) - a(x)}{2A(x)^2}\Big)^{1/2}\partial_t u\Big\|^2_{L^2L^2_{|x|<R/2}} + \frac{\delta}{2R^{4m_1+1}}\Big\|\Big(\frac{a(x)A(x) - a(x)}{A(x)^2}\Big)^{1/2}|\nabla\!\!\!/ u|\Big\|^2_{L^2L^2_{|x|<R/2}} \\
&\qquad + \frac{1}{2R}\Big\|\Big(\frac{a(x)}{A(x)^2}\Big)^{1/2}\partial_x u\Big\|^2_{L^2L^2_{|x|<R/2}} + \frac{1}{R}\Big\|\langle x\rangle^{-5/2}u\Big\|^2_{L^2L^2_{|x|<R/2}} \\
&\qquad \lesssim E[u](0) + \int_0^T \int |\Box_{\mathfrak{g}} u|\Big(|\partial u| + \langle x\rangle^{-5/2}|u|\Big)\, dV dt + \|u\|^2_{LE^1_{|x|\approx R}} \\
&\qquad + \frac{1}{R^{4m_1}}\Big\|\langle x\rangle^{-3/2}u\Big\|^2_{L^2L^2{}_{|x|<R}}.
\end{aligned} \tag{3.18}$$

Proof. We use

$$w = \beta(|x|/R)u, \qquad f = \frac{A(x) - 1}{R}, \qquad g(x) = \frac{1}{2}f'(x) + \frac{\delta}{R^{4m_1}}\frac{A'(x)}{A(x)}f(x),$$

and we shall examine the coefficients of each term on the right hand side of (3.2). Here $\delta > 0$ is a small parameter that will be fixed later.

First, for the time derivative, we have

$$-g(x) + \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big[A(x)^2 f(x)\big]\Big\} = \Big(1 - \frac{\delta}{R^{4m_1}}\Big)\frac{a(x)}{2A(x)^2}f(x) = \Big(1 - \frac{\delta}{R^{4m_1}}\Big)\frac{a(x)}{2A(x)^2}\frac{A(x) - 1}{R}.$$

We now choose $\delta > 0$ to be sufficient small so that the above coefficient is nonnegative. It thus follows that

$$\int_0^T \int \Big\{ -g(x) + \frac{1}{2}\Big(A(x)^{-2}\partial_x(A(x)^2 f(x)\Big)\Big\}(\partial_t w)^2 \, dV dt \\ \gtrsim \Big(\frac{1}{R} - \frac{\delta}{R^{4m_1+1}}\Big)\Big\|\Big(\frac{a(x)A(x)-a(x)}{2A(x)^2}\Big)^{1/2}\partial_t u\Big\|^2_{L^2 L^2_{|x|<R/2}}. \tag{3.19}$$

Here and throughout the proof, all the implicit constants in the inequalities are independent of R.

For the angular derivative, we have

$$f(x)\frac{A'(x)}{A(x)} + g(x) - \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big[A(x)^2 f(x)\big]\Big\} = \frac{\delta}{R^{4m_1}}\frac{A'(x)}{A(x)}f(x) = \frac{\delta}{R^{4m_1}}\frac{a(x)}{2A(x)^2}\frac{A(x)-1}{R},$$

which is easily seen to be nonnegative. Thus,

$$\int_0^T \int \Big\{f(x)\frac{A'(x)}{A(x)} + g(x) - \frac{1}{2}\Big(a(x)^{-2}\partial_x(a(x)^2 f(x)\Big)\Big\}\frac{1}{A(x)^2}|\nabla_0 w|^2 \, dV dt \\ \gtrsim \frac{\delta}{2R^{4m_1+1}}\Big\|\Big(\frac{a(x)A(x)-a(x)}{A(x)^2}\Big)^{1/2}|\nabla u|\Big\|^2_{L^2 L^2_{|x|<R/2}}. \tag{3.20}$$

For the coefficient of $(\partial_x w)^2$, noting that $\frac{A'(x)}{A(x)}f(x) = \frac{a(x)}{2A(x)^2}f(x) > 0$, we have

$$\begin{aligned} f'(x) + g(x) - \frac{1}{2}\Big\{A(x)^{-2}\partial_x\Big(A(x)^2 f(x)\Big)\Big\} &= \Big(\frac{\delta}{R^{4m_1}} - 1\Big)\frac{a(x)}{2A(x)^2}f(x) + f'(x) \\ &\geq f'(x) - \frac{a(x)}{2A(x)^2}f(x) \\ &= \frac{a(x)}{2RA(x)}\Big(1 - \frac{A(x)-1}{A(x)}\Big) = \frac{a(x)}{2RA(x)^2}, \end{aligned}$$

which is everywhere nonnegative. Hence,

$$\int_0^T \int \Big(f'(x) + g(x) - \frac{1}{2}\Big\{A(x)^{-2}\partial_x\big(A(x)^2 f(x)\big)\Big\}\Big)(\partial_x w)^2 dV \, dt \\ \gtrsim \frac{1}{2R}\Big\|\Big(\frac{a(x)}{A(x)^2}\Big)^{1/2}\partial_x u\Big\|^2_{L^2 L^2_{|x|<R/2}}. \tag{3.21}$$

For the lower order term, we first compute

$$-\frac{1}{4}A(x)^{-2}\partial_x\big(A(x)^2\partial_x f'(x)\big) = -\frac{a''(x)}{8A(x)R} + \frac{3a(x)a'(x)}{4A(x)^3 R} - \frac{a(x)^3}{16A(x)^5 R}.$$

On the other hand, we have

$$\begin{aligned} &-\frac{1}{2R^{4m_1}}A(x)^{-2}\partial_x\Big(A(x)^2\partial_x\Big(\frac{a(x)}{2A(x)^2}f(x)\Big) \\ =&\frac{1}{2R^{4m_1}}\Big(-\frac{a'(x)(A(x)-1)}{2A(x)^2 R} - \frac{3a(x)a'(x)}{4RA(x)^3} + \frac{a(x)a'(x)(A(x)-1)}{RA(x)^4} + \frac{3a(x)^3}{8RA(x)^5} - \frac{a(x)^3(A(x)-1)}{2RA(x)^6}\Big). \end{aligned}$$

This entire coefficient is bounded above by $\frac{1}{R^{4m_1}}\langle x\rangle^{-3}$. Since here an upper bound is needed, we are allowed to attach it to the right hand side of our estimate.

Using (3.19), (3.20), (3.21) in (3.2), we have

$$
\begin{aligned}
&-\int_0^T \int \Box_{\mathfrak{g}} w\{f(x)\partial_x w + g(x)w\}\ dV dt - \int \partial_t w(f(x)\partial_x w + g(x)w)\ dV\Big|_0^T \\
&\qquad \gtrsim \frac{1}{2R}\Big\|\Big(\frac{a(x)}{A(x)^2}\Big)^{1/2}\partial_x u\Big\|^2_{L^2L^2_{|x|<R/2}} + \frac{\delta}{2R^{4m_1+1}}\Big\|\Big(\frac{a(x)A(x)-a(x)}{A(x)^2}\Big)^{1/2}|\nabla\!\!\!/ u|\Big\|^2_{L^2L^2_{|x|<R/2}} \\
&\qquad\quad + \Big(\frac{1}{R} - \frac{\delta}{R^{4m_1+1}}\Big)\Big\|\Big(\frac{a(x)A(x)-a(x)}{2A(x)^2}\Big)^{1/2}\partial_t u\Big\|^2_{L^2L^2_{|x|<R/2}} - \frac{1}{R^{4m_1}}\Big\|\langle x\rangle^{-3/2}u\Big\|^2_{L^2L^2{}_{|x|<R}}.
\end{aligned} \tag{3.22}
$$

$f(x)$ is bounded independent of R, and $|g(x)| \lesssim 1/R$ on the support of $\beta(|x|/R)$. Thus we can again apply the Cauchy-Schwarz inequality and the Hardy inequality to bound each of the time-boundary terms in by the energy at that time, and note that the nonhomogeneous term can be controlled by $\|u\|_{LE^1_R}$ as shown in (3.17). And from what results, the desired estimate follows immediately.

□

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